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Transition from laminar convection to thermal turbulence in a rotating fluid layer

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The convective flow in an infinite horizontal fluid layer rotating rigidly about a normal axis is investigated for the special case of infinite Prandtl number and free boundary conditions. For slightly supercritical Rayleigh numbers the solutions of the non-linear steady-state equations are derived approximately by an amplitude expansion. A stability calculation shows that no stable steady-state convective flow exists if the Taylor number exceeds the critical value 2285.

1. Introduction

If a layer of heavy fluid is heated from below a density gradient opposite to the force of gravity is produced by thermal expansion. A dimensionless measure of the temperature difference ΔT between the top and bottom of the layer is the Rayleigh number

$$R = \alpha g \Delta T d^3/\nu\kappa,$$

where α is the expansion coefficient, g the acceleration of gravity, d the depth of the layer, ν the kinematic viscosity, and κ the thermometric conductivity. In cases where the Rayleigh number exceeds a certain critical value R_c the static state becomes unstable and convective motions arise.

It is well known that rotation about a normal axis with angular velocity $\mathbf{\Omega}$ has an inhibiting effect on the onset of convection, i.e. the critical Rayleigh number is raised by the rotation. Here we investigate how the rotation rate described by the Taylor number

$$\tau^2 = 4|\mathbf{\Omega}|^2 d^4/\nu^2$$

changes the stability behaviour of the steady-state convective flow.

As in the non-rotating case the steady-state cellular convective flow is not uniquely determined by the equations of motion and the boundary conditions. In fact, if the layer is of infinite horizontal extent an infinite manifold of solutions is obtained.

Solutions of the linear steady-state equations were derived by Chandrasekhar (1953). These are also given by Chandrasekhar (1961) together with a comprehensive bibliography. Veronis (1958) derived solutions of the non-linear equations by expanding with respect to an amplitude parameter and showed that hexagons, squares, and rolls are solutions of the steady-state system of equations. However, the rectangles treated by Veronis do not satisfy the non-linear equations.

To sort the physically realized solutions from the manifold obtained from the steady-state system of equations, it is necessary to make a stability calculation.

As we consider types of flow of relatively low amplitude, it is reasonable to solve the stability equations as well by successive approximation.

The ensemble of possible solutions is considered for the case of an infinite Prandtl number and 'free-free' boundary conditions. The steady-state and stability equations are solved by the method used by Schülter, Lortz & Busse (1965) in the non-rotating case.

2. Fundamental equations

With κ/d as velocity scale, $\nu\kappa/\alpha g d^3$ as temperature scale, d^2/κ as time scale and d as length scale and with the Boussinesq approximation, the conservation laws of mass, momentum and energy read in dimensionless form:

$$\partial_j u_j = 0, \tag{2.1}$$

$$P^{-1}(\partial_t u_i + u_j \partial_j u_i) = -\partial_i \Gamma + \Theta\lambda_i + \Delta u_i + \tau\epsilon_{ijk} u_j \lambda_k, \tag{2.2}$$

$$\left.\begin{aligned} \partial_t \Theta + u_j \partial_j \Theta &= R u_j \lambda_j + \Delta\Theta, \\ P = \nu/\kappa, \quad R = \alpha g \Delta T d^3/\nu\kappa, &\quad \tau = 2\Omega_j \lambda_j d^2/\nu. \end{aligned}\right\} \tag{2.3}$$

We have used the summation convention and the notation

$$\partial_t = \frac{\partial}{\partial t}, \quad \partial_j = \frac{\partial}{\partial x_j} \quad (j = 1, 2, 3).$$

λ_i is the unit vector opposite to the force of gravity. All terms that can be written as gradients are summarized by $\partial_i \Gamma$. Θ denotes the deviation from the linear temperature distribution of the static state.

In the limit of large Prandtl number P the non-linear and time-differentiated terms of the momentum equation can be neglected. It is well known that the essential features of finite amplitude convection are still present in this limit. Preliminary calculations for the case of rigid boundary conditions and large, but finite, Prandtl number yield the same qualitative results, i.e. there is a continuous dependence on the parameter P^{-1} for $P^{-1} \to 0$.

The general solution of the continuity equation (2.1) for the geometry under consideration can be written

$$\left.\begin{aligned} u_i &= \delta_i v + \epsilon_i w, \\ \delta_i = \partial_i \partial_k \lambda_k - \lambda_i \Delta, &\quad \epsilon_i = \epsilon_{ijk} \lambda_j \partial_k, \end{aligned}\right\} \tag{2.4}$$

where v and w are arbitrary functions. By applying δ_i and ϵ_i to (2·2) and by using (2.4) in (2.3) one obtains

$$\Delta^2 \Delta_2 v + \tau \partial_z \Delta_2 w - \Delta_2 \Theta = 0, \tag{2.5}$$

$$-\tau \partial_z \Delta_2 v + \Delta\Delta_2 w = 0, \tag{2.6}$$

$$-R\Delta_2 v + \Delta\Theta = \partial_t \Theta + (\delta_j v + \epsilon_j w)\partial_j \Theta, \tag{2.7}$$

$$\Delta_2 = \Delta - \partial^2_{zz}, \quad \partial_z = \partial_j \lambda_j.$$

To simplify the notation we introduce the matrix differential operators

$$U = \begin{pmatrix} \Delta^2\Delta_2 & \tau\partial_z\Delta_2 & -\Delta_2 \\ -\tau\partial_z\Delta_2 & \Delta\Delta_2 & 0 \\ 0 & 0 & \Delta \end{pmatrix},$$

$$W = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\Delta_2 & 0 & 0 \end{pmatrix},$$

the three-component column matrices

$$X = \begin{pmatrix} v \\ w \\ \Theta \end{pmatrix}, \quad Q(X', X) = \begin{pmatrix} 0 \\ 0 \\ (\delta_j v' + \epsilon_j w')\,\partial_j \Theta \end{pmatrix},$$

and the matrix

$$V = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and obtain

$$RWX + UX = Q(X, X) + \partial_t VX. \tag{2.8}$$

The steady-state system then has the form

$$RWX + UX = Q(X, X). \tag{2.9}$$

The stability of the solutions X of the system (2.9) is investigated by superposing on the solutions X non-stationary perturbations $\tilde{X}$ of infinitesimal amplitude and with the time dependence $e^{\sigma t}$. This yields the linear stability equations

$$RW\tilde{X} + U\tilde{X} = \sigma V\tilde{X} + Q(X, \tilde{X}) + Q(\tilde{X}, X). \tag{2.10}$$

X is unstable if the system (2.10) with boundary conditions has solutions with a positive real part of σ.

3. Boundary conditions

The layer is of infinite horizontal extent and all functions occurring are everywhere bounded. Suppose that at $z = \pm\frac{1}{2}$ the layer is bounded by a perfectly conducting medium. This corresponds to the boundary condition

$$\Theta = 0, \quad z = \pm\tfrac{1}{2}, \tag{3.1}$$

for the temperature. Since there ought to be no shear stress in the free-free case one obtains the following dynamical boundary conditions

$$v = \partial^2_{zz} v = \partial_z w = 0, \quad z = \pm\tfrac{1}{2}. \tag{3.2}$$

4. Small amplitude perturbation theory

Steady-state solutions of the non-linear system (2.9) and solutions of the stability equations (2.10) are sought by expanding the equations for small amplitude of the steady-state solutions. This is equivalent to treating the convection near $R = R_c$. We therefore expand X and R with respect to an amplitude parameter ϵ as follows:

$$X = \epsilon X_1 + \epsilon^2 X_2 + \epsilon^3 X_3 + \ldots, \tag{4.1}$$

$$R = R_0 + \epsilon R_1 + \epsilon^2 R_2 + \ldots. \tag{4.2}$$

Substitution of these in the system (2.9) yields a set of inhomogeneous equations the solubility conditions of which determine the R_n. As R is an externally given parameter, (4.2) can be used for determining ϵ.

The stability equations are solved by expanding also σ and $\tilde{X}$ with respect to ϵ.

$$\tilde{X} = \tilde{X}_1 + \epsilon\tilde{X}_2 + \epsilon^2\tilde{X}_3 + \dots, \tag{4.3}$$

$$\sigma = \sigma_0 + \epsilon\sigma_1 + \epsilon^2\sigma_2 + \dots. \tag{4.4}$$

5. Solution of the linear problem

$$R_0\, WX_1 + UX_1 = 0. \tag{5.1}$$

$$R_0\, W\tilde{X}_1 + U\tilde{X}_1 = \sigma_0 V\tilde{X}_1. \tag{5.2}$$

The solutions of these equations with the boundary conditions in § 3 are well-known (see Chandrasekhar (1961) and Schlüter *et al.* (1965)). The solution of (5.1) can be written in the form:

$$\left.\begin{aligned} X_1 &= \begin{pmatrix} v_1 \\ w_1 \\ \Theta_1 \end{pmatrix} = \begin{pmatrix} f(z) \\ g(z) \\ h(z) \end{pmatrix} \sum_{m=-N}^{+N} C_m \omega_m, \\ \omega_m &= \exp(i\mathbf{k}_m.\mathbf{r}), \quad |\mathbf{k}_m|^2 = a^2, \\ f &= \cos\pi z, \quad g = -Q\,df/dz, \quad h = Sf, \\ S &= R_0 a^2/(\pi^2+a^2), \quad Q = \tau/(\pi^2+a^2). \end{aligned}\right\} \tag{5.3}$$

$\mathbf{r}$ is a horizontal position vector and $\mathbf{k}_m$ a horizontal wave-number vector with overall wave-number a. R_0 is given by

$$R_0 = [(\pi^2+a^2)^3 + \pi^2\tau^2]/a^2. \tag{5.4}$$

R_0 attains its minimum R_c for values a_c^2 satisfying the equation

$$2x^3 + 3x^2 = 1 + \tau^2/\pi^4, \quad x = a_c^2/\pi^2. \tag{5.5}$$

In the following we put $a = a_c$ because this is the only case where (5.2) yields no instability. Then the most critical disturbance has $\sigma_0 = 0$ and is described by

$$\tilde{X}_1 = \begin{pmatrix} \tilde{v}_1 \\ \tilde{w}_1 \\ \tilde{\theta}_1 \end{pmatrix} = \begin{pmatrix} f \\ g \\ h \end{pmatrix} \sum_m \tilde{C}_m \omega_m. \tag{5.6}$$

We now define a scalar product

$$\langle X', X\rangle = R_0(v'v)_m + R_0(w'w)_m + (\Theta'\Theta)_m$$

with functions v, v', w, w', Θ, and Θ' satisfying the boundary conditions. $(\ )_m$ denotes averaging over the layer. The operator $U + R_0\, W$ is then self-adjoint in the following sense

$$\langle X', (U+R_0\, W)\, X\rangle = \langle X, (U+R_0\, W)\, X'\rangle. \tag{5.7}$$

6. Second-order solutions

On forming the scalar product with the inhomogeneous second-order equation

$$R_0 WX_2 + UX_2 = Q(X_1, X_1) - R_1 WX_1 \tag{6.1}$$

and an arbitrary first-order solution X_1' and using the self-adjointness relation (5.7), one obtains the solubility condition

$$\left.\begin{aligned} 0 &= \langle X_1', Q(X_1, X_1)\rangle - R_1\langle X_1', WX_1\rangle, \\ X_1' &= \begin{pmatrix} f \\ g \\ h \end{pmatrix} \omega_n \quad (n = -\infty, \ldots, +\infty). \end{aligned}\right\} \tag{6.2}$$

Because of the symmetry of the function $f(z)$ it is readily seen that the term $\langle X_1', Q(X_1, X_1)\rangle$ is equal to zero. This gives

$$0 = R_1\langle X_1', WX_1\rangle = -R_1(\Theta_1' \Delta_2 v_1)_m = -R_1(\Theta_1'\Delta\Theta_1)_m/R_0 = R_1((\partial_j \Theta_1')\partial_j\Theta_1)_m/R_0$$

and from this it follows that $R_1 = 0$.

For $R_1 = 0$ the stability equations in the second-order are

$$R_0 W\tilde{X}_2 + U\tilde{X}_2 = Q(\tilde{X}_1, X_1) + Q(X_1, \tilde{X}_1) + \sigma_1 V\tilde{X}_1, \tag{6.3}$$

which yield the solubility condition

$$0 = \langle X_1', Q(\tilde{X}_1, X_1)\rangle + \langle X_1', Q(X_1, \tilde{X}_1)\rangle + \sigma_1\langle X_1', V\tilde{X}_1\rangle. \tag{6.4}$$

The first two triple products on the right-hand side are again zero and it follows that

$$\sigma_1 = 0.$$

To determine R_2 and σ_2 we calculate the solutions X_2 and $\tilde{X}_2$. The equations for v_2, w_2, and Θ_2 are

$$\left.\begin{aligned} (\Delta^3 + \tau^2\partial_{zz}^2 - R_0\Delta_2)\, v_2 &= (\delta_j v_1 + \epsilon_j w_1)\partial_j\Theta_1, \\ (\Delta^3 + \tau^2\partial_{zz}^2 - R_0\Delta_2)\Delta w_2 &= \tau\partial_z(\delta_j v_1 + \epsilon_j w_1)\partial_j\Theta_1, \\ (\Delta^3 + \tau^2\partial_{zz}^2 - R_0\Delta_2)\Delta\Theta_2 &= (\Delta^3 + \tau^2\partial_{zz}^2)(\delta_j v_1 + \epsilon_j w_1)\partial_j\Theta_1, \end{aligned}\right\} \tag{6.5}$$

with boundary conditions:

$$\begin{aligned} v_2 = D^2 v_2 = D^4 v_2 = 0, \quad D = \partial_z, \\ Dw_2 = D^3 w_2 = D^5 w_2 = D^7 w_2 = 0, \\ \Theta_2 = D^2\Theta_2 = D^4\Theta_2 = D^6\Theta_2 = 0. \end{aligned}$$

We now try to find X_2 in the form

$$\left.\begin{aligned} X_2 &= \sum_{k,l=-N}^{+N} \begin{pmatrix} F(\phi_{kl}, z) \\ G(\phi_{kl}, z) \\ H(\phi_{kl}, z) \end{pmatrix} C_k C_l \omega_k \omega_l, \\ \phi_{kl} &= \mathbf{k}_k . \mathbf{k}_l / a^2. \end{aligned}\right\} \tag{6.6}$$

The z-dependences of the inhomogeneities of the equations (6.5) are proportional to $\sin 2\pi z$ or $\cos 2\pi z$. One therefore gets a solution satisfying the boundary conditions, if the expressions F, G, and H are of the form

$$\left.\begin{aligned} F(\phi_{kl}, z) &= -\tfrac{1}{2}\pi a^2 S(1-\phi_{kl}) \sin 2\pi z / D(\phi_{kl}), \\ G(\phi_{kl}, z) &= \frac{\tau \pi^2 a^2 S(1-\phi_{kl}) \cos 2\pi z}{[4\pi^2 + 2a^2(1+\phi_{kl})]\, D(\phi_{kl})}, \\ H(\phi_{kl}, z) &= \frac{\{[4\pi^2 + 2a^2(1+\phi_{kl})]^3 + 4\pi^2\tau^2\}\, F(\phi_{kl}, z)}{4\pi^2 + 2a^2(1+\phi_{kl})}, \end{aligned}\right\} \tag{6.7}$$

$$D(\phi_{kl}) = -[4\pi^2 + 2a^2(1+\phi_{kl})]^3 - 4\pi^2\tau^2 + 2a^2 R_0(1+\phi_{kl}).$$

Because of

$$\sum_{kl} F(\phi_{kl}, z)\,(C_k \tilde{C}_l + \tilde{C}_k C_l)\,\omega_k \omega_l = 2 \sum_{kl} F(\phi_{kl}, z)\, C_k \tilde{C}_l \omega_k \omega_l, \quad \text{etc.}$$

the analogous form for $\tilde{X}_2$ is

$$\tilde{X}_2 = 2 \sum_{k=-N}^{+N} \sum_{l=-\infty}^{+\infty} \begin{pmatrix} F(\phi_{kl}, z) \\ G(\phi_{kl}, z) \\ H(\phi_{kl}, z) \end{pmatrix} C_k \tilde{C}_l \omega_k \omega_l. \tag{6.8}$$

7. Order ϵ^3

The system of equations is

$$R_0 W X_3 + U X_3 = Q(X_1, X_2) + Q(X_2, X_1) - R_2 W X_1, \tag{7.1}$$

$$R_0 W \tilde{X}_3 + U \tilde{X}_3 = Q(\tilde{X}_1, X_2) + Q(X_1, \tilde{X}_2) + Q(\tilde{X}_2, X_1) + Q(X_2, \tilde{X}_1) - R_2 W \tilde{X}_1 + \sigma_2 V \tilde{X}_1. \tag{7.2}$$

The solubility conditions, which postulate that the right-hand sides of (7.1) and (7.2) be orthogonal to all first-order solutions X_1', are of the form

$$0 = \langle X_1', Q(X_1, X_2) + Q(X_2, X_1)\rangle - R_2 \langle X_1', W X_1\rangle, \tag{7.3}$$

$$0 = \langle X_1', Q(\tilde{X}_1, X_2) + Q(X_1, \tilde{X}_2) + Q(\tilde{X}_2, X_1) + Q(X_2, \tilde{X}_1)\rangle - R_2 \langle X_1', W \tilde{X}_1\rangle + \sigma_2 \langle X_1', V \tilde{X}_1\rangle. \tag{7.4}$$

Before studying these equations more closely we transform the first term of the right-hand side of (7.3) by partial integration:

$$\langle X_1', Q(X_1, X_2)\rangle = -(\Theta_2 \partial_j \Theta_1' (\delta_j v_1 + \epsilon_j w_1))_m,$$

$$\langle X_1', Q(X_2, X_1)\rangle = (v_2 \delta_j (\Theta_1' \partial_j \Theta_1) - w_2 \epsilon_j \Theta_1' \partial_j \Theta_1)_m,$$

$$\delta_j \Theta_1' \partial_j \Theta_1 = 0 \quad \text{(see Schlüter } et\ al. \text{ (1965)).}$$

Substituting this in (7.3) and taking into account the corresponding expressions for the first-order and second-order functions, one obtains with

$$X_1' = \begin{pmatrix} f \\ g \\ h \end{pmatrix} \omega_n^* \quad (n = -\infty, \ldots, +\infty),$$

$$0 = a^2R_2 \sum_{m=-N}^{+N} (fhC_m\omega_m\omega_n^*)_m + \sum_{k,l,m=-N}^{+N} L_{klmn}C_kC_lC_m(\omega_k\omega_l\omega_m\omega_n^*)_m, \tag{7.5}$$

$$L_{klmn} = (H(\phi_{kl},z)a^2S[(1+\phi_{mn})-Q\psi_{mn}]f'f - G(\phi_{kl},z)a^2Sf^2\psi_{mn})_m, \tag{7.5a}$$

$$\psi_{mn} = (\mathbf{k}_m \times \mathbf{k}_n).\boldsymbol{\lambda}/a^2.$$

Equation (7.4) is similarly transformed, giving

$$0 = \sigma_2 \sum_m \tilde{C}_m(\omega_m\omega_n^*h^2)_m - a^2R_2\sum_m \tilde{C}_m(\omega_m\omega_n^*fh)_m - \sum_{k,l,m} C_kC_l\tilde{C}_m(L_{klnm}+2L_{kmnl})(\omega_k\omega_l\omega_m\omega_n^*)_m. \tag{7.6}$$

The horizontal averaging only makes a contribution if the sum of the four $\mathbf{k}$-vectors becomes zero, i.e. if it holds that: (i) $k = n, l = -m$; (ii) $l = n, k = -m, k \neq n$; (iii) $m = n, k = -l, k \neq n, k \neq -n$.

For further simplification we introduce the following matrix T_{ik}:

$$T_{ik} = \begin{cases} -2L_{i,-i,i,i} & \text{for} \quad i = \pm k, \\ -2(L_{i,-k,ki} + L_{ik,-k,i} + L_{-i,ikk}) & \text{otherwise.} \end{cases} \tag{7.7}$$

According to the definition of L this has the symmetry properties

$$T_{ik} = T_{-i,k} = T_{i,-k}. \tag{7.8}$$

In contrast to the non-rotating case it should be observed that T_{ik} is not symmetric. Note that all diagonal elements are equal. Equations (7.5) can be written as follows:

$$\left.\begin{aligned} R_2K &= \frac{1}{2}\sum_{k=-N}^{+N} T_{ik}|C_k|^2 \quad (i = -N, \ldots, -1, +1, \ldots, +N), \\ K &= a^2(fh)_m. \end{aligned}\right\} \tag{7.9}$$

From the symmetry properties of T_{ik} it follows that only N equations are independent and together with the normalization condition

$$\sum_{k=1}^{N} |C_k|^2 = \tfrac{1}{2} \tag{7.10}$$

we obtain an inhomogeneous system of $N+1$ equations for determining R_2 and $|C_k|^2$. This means that the manifold of first-order solutions is restricted by the non-linear terms of the equations.

8. Discussion of the steady-state solubility condition

For $N = 1$, (7.9) and (7.10) read

$$|C_1|^2 = \tfrac{1}{2}, \quad R_2K = \tfrac{1}{2}T_{11}.$$

For $N = 2$, i.e. for rectangles and squares, it is found by elimination from (7.9) and (7.10) that

$$|C_1|^2 = \frac{1}{2}\frac{T_{12}-T_{11}}{T_{12}+T_{21}-2T_{11}}, \quad |C_2|^2 = \frac{1}{2}\frac{T_{21}-T_{11}}{T_{12}+T_{21}-2T_{11}}, \tag{8.1}$$

$$R_2K = \sum_{k=1}^{2} |C_k|^2 T_{1k}, \tag{8.2}$$

where $T_{11} = T_{22}$ has been used. The condition for the existence of solutions with $|C_1|^2 = |C_2|^2$ considered by Veronis (1958) is that $T_{12} = T_{21}$. This, however, is only the case for squares and limiting rectangles.

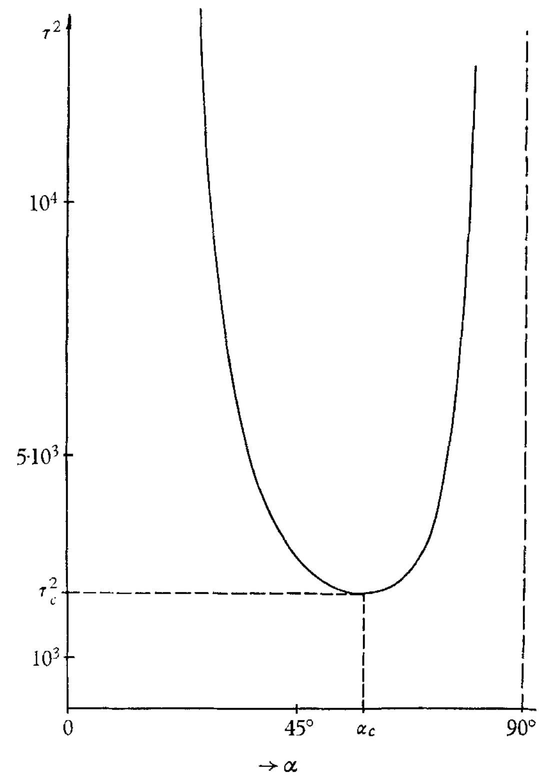

FIGURE 1. τ^2, Taylor number; α, angle between the two **k**-vectors. In the convex domain rectangles do not exist and rolls are unstable.

Furthermore, one can see that rectangular solutions can only exist if $T_{12} - T_{11}$ and $T_{21} - T_{11}$ have the same sign. Figure 1 shows the region in which this condition is violated. Certain Taylor numbers thus disallow certain patterns of rectangles characterized by the angle α. For $\tau \rightarrow \infty$ squares and limiting rectangles are the only remaining possibilities.

If $N \geqslant 3$ one obtains in the regular case, i.e. if the angles formed by adjacent **k**-vectors are equal, the solutions

$$|C_1|^2 = \ldots = |C_N|^2 = \frac{1}{2N},$$

$$R_2 K = \frac{1}{2N} \sum_{k=1}^{N} T_{1k}.$$

9. Eigenvalue σ_2

To obtain information about the stability of the steady-state solutions not excluded by (7.9) and (7.10), we investigate (7.6).

Disturbances coincident with the basic vectors of the cell pattern

First we consider disturbances with $\tilde{C}_k = 0$ except in the case of **k**-vectors for which $C_i \neq 0$ is valid. By transforming (7.6) in the same way as (7.5) the following system of equations is derived:

$$\left.\begin{aligned} M\sigma_2 \tilde{C}_i + \sum_{k=-N}^{+N} T_{ik} C_i C_k^* \tilde{C}_k = 0 \quad (i = -N, \ldots, -1, +1, \ldots, +N), \\ M = (h^2)_m. \end{aligned}\right\} \tag{9.1}$$

This homogeneous system of equations for the $\tilde{C}_i$ only has a non-trivial solution if the determinant of the matrix of the coefficients $\tilde{C}_i$ vanishes.

$$\det |M\sigma_2 \delta_{ik} + T_{ik} C_i C_k^*| = 0 \quad (i, k = -N, \ldots, +N)$$

or

$$\det \left| \frac{M}{|C_i|^2} \sigma_2 \delta_{ik} + T_{ik} \right| = 0 \quad (i, k = -N, \ldots, +N). \tag{9.2}$$

If we subtract the $(-k)$th column from the kth and add the ith row to the $(-i)$th and use the symmetry properties of T_{ik}, it is found that N eigenvalues σ_2 are equal to zero. The remaining eigenvalues satisfy the equation

$$\det |M\sigma_2 \delta_{ik} + 2T_{ik} |C_k|^2| = 0 \quad (i, k = 1, \ldots, N). \tag{9.3}$$

Since T_{ik} is generally not symmetric one expects complex eigenvalues σ_2. Let S_{ik} be the symmetric and A_{ik} the antisymmetric part of T_{ik},

$$T_{ik} = S_{ik} + A_{ik}, \quad S_{ik} = S_{ki}, A_{ik} = -A_{ki}. \tag{9.4}$$

We prove that

$$S_{ik} > T_{ii} > 0 \quad (k \neq i) \tag{9.5}$$

is valid, the consequence of which, as we shall see, is that all three-dimensional flows are unstable.

The second part of the inequality (9.5) follows direct from the definition of L in (7.5*a*)

$$T_{ii} = -2L_{-i,iii} = \tfrac{1}{4}a^4 S^2 > 0.$$

For proving the inequality $S_{ik} > T_{ii}$ one has to compute the sign of $D(\phi_{kl})$ occurring in the expression for L.

Eliminating τ^2 and R_0 with the aid of (5.5) and (5.4), respectively, and arranging in powers of a^2 we find for $D(\phi_{kl})$ defined in (6.7):

$$D(\phi_{kl}) = -2a^6(4\alpha_{kl}^3 + 4 - 3\alpha_{kl}) - 12a^4\pi^2(4\alpha_{kl}^2 - \alpha_{kl} + 1) - 90a^2\pi^4\alpha_{kl} - 60\pi^6 < 0$$

for

$$0 \leqslant \alpha_{kl} \leqslant 2, \quad \alpha_{kl} = 1 + \phi_{kl}.$$

Thus

$$L_{klmn} + L_{klnm} = \frac{\pi^2 a^4 S^2 \{[4\pi^2 + 2a^2(1+\phi_{kl})]^3 + 4\pi^2\tau^2\}}{4D(\phi_{kl})\,[4\pi^2 + 2a^2(1+\phi_{kl})]} (1-\phi_{kl})(1+\phi_{mn}) < 0$$

for

$$k \neq \pm l, \quad m \neq \pm n$$

and consequently

$$S_{ik} - T_{ii} = -(L_{i,-k,ki} + L_{k,-i,ik} + L_{ik,-k,i} + L_{ki,-i,k}) > 0 \quad \text{for} \quad k \neq i.$$

It will now be shown that for $N > 1$ relation (9.5) implies the existence of an eigenvalue σ_2 with positive real part. Adding all columns of the determinant (9.3) and taking (7.9) into account yields

$$\left.\begin{aligned} (2R_2K + \sigma_2 M) \det |B_{ik} + \sigma_2 M\delta_{ik}| &= 0, \\ B_{ik} = 2(T_{ik} - T_{Nk}) |C_k|^2, \quad (i, k &= 1, \dots, N-1). \end{aligned}\right\} \tag{9.6}$$

Equation (9.6) is satisfied if:

$$\text{(i)} \quad \sigma_2 = -2R_2K/M, \tag{9.7}$$

or

$$\text{(ii)} \quad \det |B_{ik} + \sigma_2 M\delta_{ik}| = 0 \quad (i, k = 1, \dots, N-1) \tag{9.8}$$

is valid. We form the trace tr of the matrix B_{ik}

$$\begin{aligned} \text{tr} &= \sum_{i=1}^{N-1} B_{ii} = \sum_{i=1}^{N-1} 2(T_{ii} - T_{Ni}) |C_i|^2 \\ &= 2T_{ii} \sum_{i=1}^{N-1} |C_i|^2 - 2R_2K + 2T_{NN} |C_N|^2 = T_{ii} - 2R_2K. \end{aligned} \tag{9.9}$$

Here (7.9) and (7.10) were again used. Multiplying the ith equation in the system (7.9) by $|C_i|^2$ and then adding all the equations gives, because of (7.10),

$$R_2K = 2 \sum_{i,k=1}^{N} |C_i|^2 |C_k|^2 T_{ik} = 2 \sum_{i,k=1}^{N} |C_i|^2 |C_k|^2 S_{ik}. \tag{9.10}$$

Substituting this in (9.9) and using the inequality (9.5) yields

$$\text{tr} < T_{ii} - 4T_{ii} \sum_{i,k=1}^{N} |C_i|^2 |C_2|^2 = 0$$

which indicates that the sum of the roots of the real equation (9.8) is positive.

Thus it has been shown that for $N \geqslant 2$ there always exists an eigenvalue σ_2 the real part of which is larger than zero. For rolls ($N = 1$) one gets the only eigenvalue

$$\sigma_2 = -2R_2K/M = -T_{ii}/M < 0,$$

i.e. these are stable with respect to disturbances coincident with the pattern.

Disturbances not coincident with the basic vector of a roll

Before one can comment on the stability of the roll solutions, one must consider disturbances the **k**-vectors of which are not coincident with those of the steady-state pattern. These produce only diagonal elements in (7.6) and yield the continuous eigenvalue

$$\sigma_2 M = L_{1,r,-1,r} + L_{-1,r,1,r} = -\tfrac{1}{2}(T_{r1} - T_{11}).$$

So we see that the respective neutral curve τ^2 versus α coincides with the curve plotted in figure 1 where α is now the angle formed by the disturbance **k**-vector $\mathbf{k}_r$ and the basic steady-state **k**-vector $\mathbf{k}_1$. The critical values are

$$\tau_c^2 = 2285{\cdot}0 \pm 0{\cdot}1, \quad \alpha_c = 58^\circ \pm 0{\cdot}5^\circ.$$

Without numerical computation it can be seen that such a critical Taylor number must exist. For asymptotically large τ, that part of L_{klmn} which is proportional to $\psi_{mn} = \sin\alpha$ varies as $\tau^{\frac{11}{3}}$, while the other part proportional to $\phi_{mn} = \cos\alpha$ varies as $\tau^{\frac{10}{3}}$. Thus, for asymptotically large τ, $\sigma_2(\alpha)$ can have either sign.

As in the non-rotating case, we show that for subcritical Taylor numbers Malkus's (1954*a, b*) principle of maximum heat transport is valid. According to inequality (9.5) and (9.10)

$$R_2 K = 2 \sum_{i,k=1}^{N} |C_i|^2 |C_k|^2 S_{ik} \geqslant 2T_{ii} \sum_{i,k=1}^{N} |C_i|^2 |C_k|^2 = \tfrac{1}{2}T_{ii},$$

where the equality sign holds only for $N = 1$, the case of rolls. Thus the only stable flow in the form of rolls possesses absolute minimum R_2, i.e.its amplitude and therefore its heat transport has an absolute maximum for a certain Rayleigh number.

10. Conclusions

It has been shown that in the rotating case the convective flow in the form of a rectangle does not exist for sufficiently large Taylor numbers. Furthermore, if the general rectangle is considered as a superposition of two rolls then the two respective amplitudes have to be unequal, except in the case of a square or a limiting rectangle.

The stability calculation yields the following results: (i) there exist stable small-amplitude rolls for slightly supercritical Rayleigh number provided $\tau^2 < \tau_c^2$. All three-dimensional convective flows are unstable; (ii) for $\tau^2 > \tau_c^2$ no stable steady-state convective flow exists in the rotating frame if the Rayleigh number is slightly supercritical, i.e. all flows are necessarily time-dependent.

For the case of infinite Prandtl number considered here it is known that for a given τ there are no steady-state solutions if R is less than $R_c(\tau)$. Thus, for $\tau^2 > \tau_c^2$ there is no stable steady-state solution at all except in the trivial static state. This means: if for a certain supercritical Taylor number the Rayleigh number is increased from a subcritical value, then there is a transition from pure conduction to a time-dependent convective flow.

For subcritical Taylor numbers the only stable flow in the form of rolls has maximum amplitude for a certain Rayleigh number and thus Malkus's hypothesis of maximum heat transport has been proved in this case. If this principle has general validity, then of all possible solutions the time-dependent flow should transport most heat for supercritical Taylor numbers. However, this interesting question has not yet been answered.

Calculations for the more realistic case of finite Prandtl number and rigid boundary conditions are in progress.

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